I. A Preview of Calculus (2.1)

The concept of a **Limit** is **central to all of calculus**. Calculus developed from finding solutions to two main problems: the **Tangent Problem** (leading to Differential Calculus) and the **Area Problem** (leading to Integral Calculus).

A. The Tangent Problem and Differential Calculus

Core Concept: Measuring the **rate of change** of a nonlinear function.

Concept	Definition / Formula	Example / Note	Citation
Secant Line	A line passing through two points $P(a, f(a))$ and $Q(x, f(x))$ on the graph of f.		
Slope of a Secant Line (m_{sec}**)**	$m_{\text{sec}} = \frac{f(x)-f(a)}{x-a}$. This formula estimates the rate of change of the function at a.	**Example 2.1:** For $f(x) = x^2$ at $P(2, 4)$, the slope using $Q(2.5, 6.25)$ is $m_{\text{sec}} = 4.5$.	
Tangent Line	The line that the secant lines approach as x approaches a. The slope of this line measures the **rate of change** (the **derivative**) of the function at a.	Solving the Tangent Problem involves taking a limit.	
Instantaneous Velocity	The value that the average velocities approach as the time interval approaches zero. It is the rate of change of the position function $s(t)$.	**Example 2.2:** A rock dropped from 64 ft with height $s(t) = 64 - 16t^2$. Average velocity approaching $t = 2$ is between -15.84 and -16.16 ft/sec (Guess: -16 ft/sec).	

Reminder: The process of letting x or t approach a in an expression is called **taking a limit**.

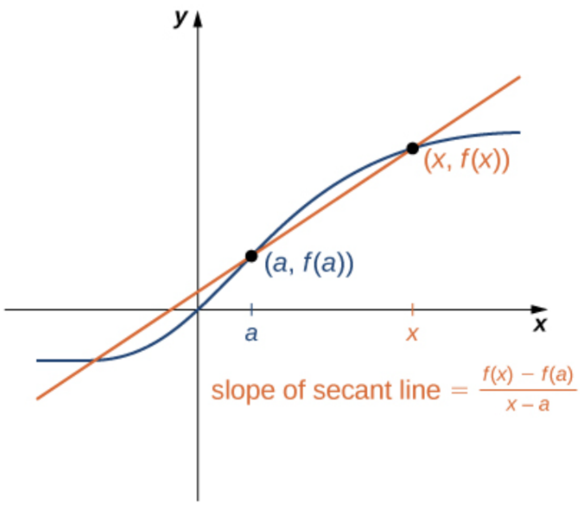


Figure 2.4 The slope of a secant line through a point $(a,\ f(a))$ estimates the rate of change of the function at the point $(a,\ f(a))$.

Check the demo:

https://calc1.drbin.top/outputs/secant_to_tangent.gif

B. The Area Problem and Integral Calculus

Core Concept: Finding the area between a function's graph and the x-axis over an interval.

Approximation: The area is approximated by dividing the interval into thin **rectangles** and summing their areas.

Integral Calculus: As the widths of the rectangles become smaller (approach zero), the sums of the areas approach the exact area. Limits of this type serve as the basis for the definition of the **definite integral**.

Example 2.3: Estimating the area under $f(x) = x^2 + 1$ over $[0, 3]$ using three rectangles yields an area estimate of $1 + 2 + 5 = 8 \text{ units}^2$.

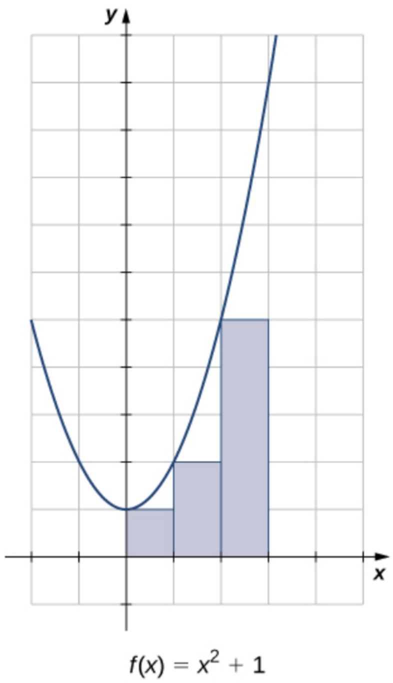
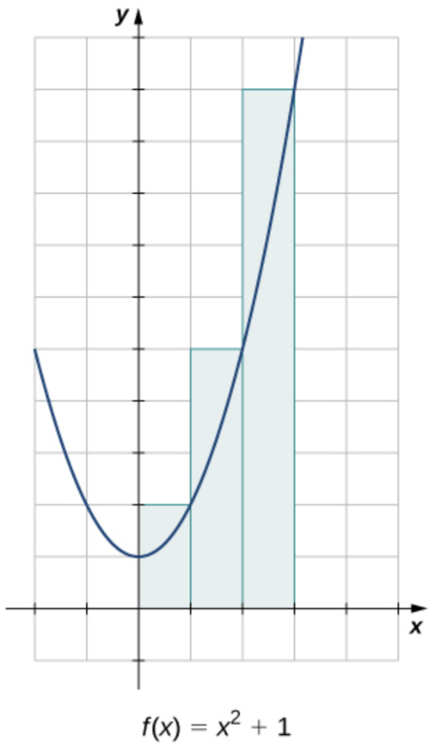


Figure 2.10 The area of the region under the curve of $f(x) = x^2 + 1$ can be estimated using rectangles.



II. The Limit of a Function (2.2)

A. Intuitive Definition of a Limit

Definition: Let f be defined in an open interval containing a, possibly excluding a itself, and let L be a real number. If all values of $f(x)$ approach the **single real number** L as the x-values approach a, then the limit of $f(x)$ as x approaches a is L.
Notation: $\lim_{x \to a} f(x) = L$.

Basic Limits (Theorem 2.1):

1. $\lim_{x \to a} x = a$.
2. $\lim_{x \to a} c = c$ (where c is a constant).

Example (Graphical Evaluation): Example 2.6: For the function shown in Figure 2.15 (where $f(-1)$ has a defined value not on the curve), as x approaches -1 from either side, the values $f(x)$ approach 3. Thus, $\lim_{x \to -1} f(x) = 3$.

Evaluating a Limit Using a Graph

For $g(x)$ shown in Figure 2.15, evaluate $\lim_{x \to -1} g(x)$.

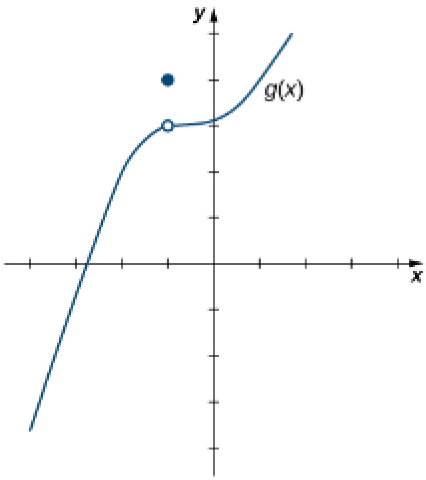


Figure 2.15 The graph of $g(x)$ includes one value not on a smooth curve.

Reminder: The limit can exist even if the function value $f(a)$ is undefined, or if $f(a)$ exists but is different from L.

B. Limits That Fail to Exist (DNE)

For the limit of a function to exist, the functional values must approach a **single real-number value** at that point.

Example (Oscillation): Example 2.7: For $f(x) = \sin(1/x)$, as x approaches 0, the y-values do not approach any single value; they **oscillate wildly between** -1 **and** 1. Therefore, $\lim_{x\to 0} \sin(1/x)$ does not exist (DNE).

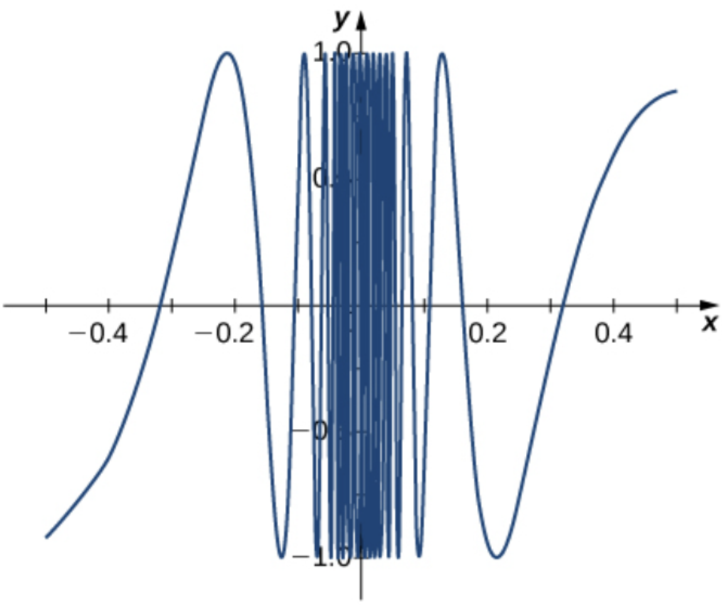


Figure 2.17 The graph of $f(x) = \sin(1/x)$ oscillates rapidly between -1 and 1 as x approaches 0.

C. One-Sided Limits

One-sided limits provide a more accurate description of function behavior, especially when a two-sided limit fails to exist.

Type	Notation	Description	Citation
Limit from the Left	$\lim_{x \to a^-} f(x) = L$	x approaches a from values $x < a$.	
Limit from the Right	$\lim_{x \to a^+} f(x) = L$	x approaches a from values $x > a$.	

Theorem 2.2: Relating One-Sided and Two-Sided Limits $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$.

Example (Jump Discontinuity): Example 2.8: For the piecewise function $f(x)$ near $x = 2$ (Figure 2.18): $\lim_{x \to 2^-} f(x) = 3$. $\lim_{x \to 2^+} f(x) = 0$. Since the one-sided limits are different, the two-sided limit $\lim_{x \to 2} f(x)$ DNE.

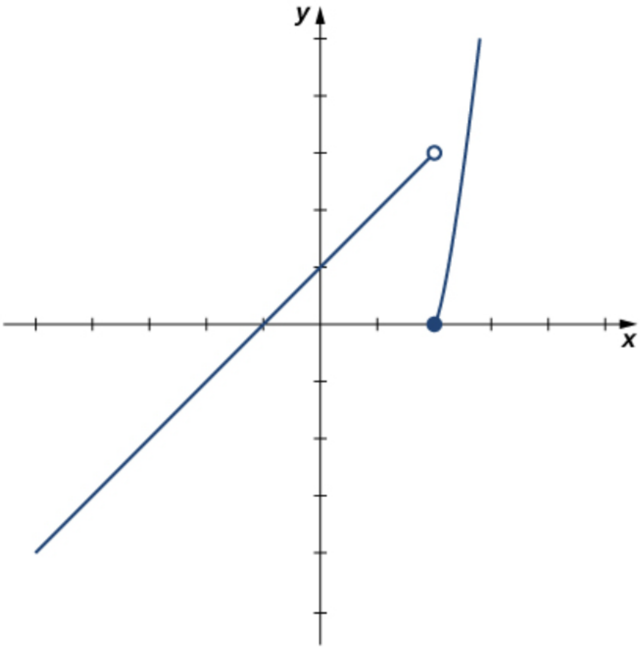


Figure 2.18 The graph of $f(x) = \begin{cases} x + 1 \text{ if } x < 2 \\ x^2 - 4 \text{ if } x \geq 2 \end{cases}$ has a break at $x = 2$.

D. Infinite Limits and Vertical Asymptotes (VA)

Definition (Infinite Limit): A function has an infinite limit at a if its values increase or decrease **without bound** as x approaches a.

If $f(x)$ increases without bound: $\lim_{x \to a} f(x) = \infty$. If $f(x)$ decreases without bound: $\lim_{x \to a} f(x) = -\infty$.

Warning: When we write $\lim_{x \to a} f(x) = \infty$, we are **describing the behavior** of the function, **not asserting that a limit exists**. If the limit is infinite, we always write ∞ (or $-\infty$) rather than DNE.

Vertical Asymptote (Definition): The line $x = a$ is a **vertical asymptote** of the graph of f if any one-sided or two-sided limit of $f(x)$ as x approaches a is infinite.

Example (VA): Example 2.9 (and Figure 2.19): For $f(x) = 1/x$: $\lim_{x \to 0^-} (1/x) = -\infty$. $\lim_{x \to 0^+} (1/x) = \infty$. The line $x = 0$ is a vertical asymptote.

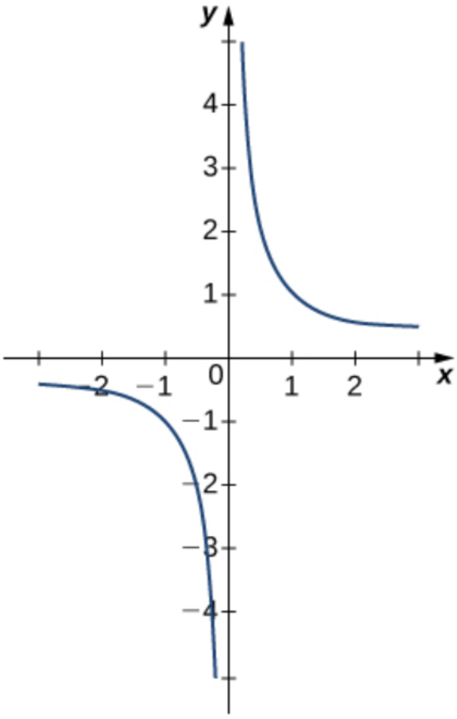


Figure 2.19 The graph of $f(x) = 1/x$ confirms that the limit as x approaches 0 does not exist.

III. The Limit Laws (2.3)

Limit laws simplify the algebraic calculation of limits.

Theorem 2.5: Limit Laws (Assuming $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, with L, M being real numbers):

Law	Formula	Condition
Sum Law	$\lim_{x \to a}(f(x) + g(x)) = L + M$	
Difference Law	$\lim_{x \to a}(f(x) - g(x)) = L - M$	
Product Law	$\lim_{x \to a}(f(x)g(x)) = LM$	
Quotient Law	$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}$	$M \neq 0$
Power Law	$\lim_{x \to a}[f(x)]^n = L^n$	n is a positive integer

A. Limits of Polynomial and Rational Functions

Theorem 2.6 (Direct Substitution): Polynomials $p(x)$**:** $\lim_{x \to a} p(x) = p(a)$.

Rational Functions $r(x) = p(x)/q(x)$**:** If a is in the domain ($q(a) \neq 0$), then $\lim_{x \to a} r(x) = r(a)$.

Example 2.16: $\lim_{x \to 3} \frac{x^2 - 6x + 5}{x^2 - 2}$.

Since $x = 3$ is in the domain, we substitute: $\frac{3^2 - 6(3) + 5}{3^2 - 2} = \frac{9 - 18 + 5}{7} = -\frac{4}{7}$.

B. Additional Techniques for the Indeterminate Form $0/0$

If direct substitution yields $\frac{0}{0}$, the limit may still exist, and we must find a function $g(x)$ identical to $f(x)$ everywhere except at a.

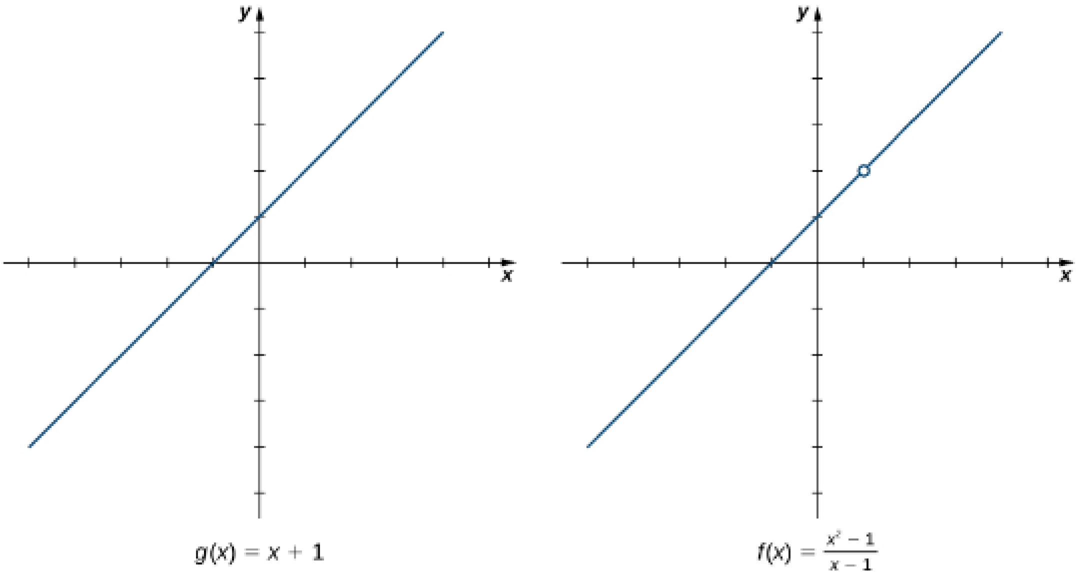


Figure 2.24 The graphs of $f(x)$ and $g(x)$ are identical for all $x \neq 1$. Their limits at 1 are equal.

We see that

$$\lim_{x\to 1}\frac{x^2-1}{x-1}=\lim_{x\to 1}\frac{(x-1)(x+1)}{x-1}=\lim_{x\to 1}(x+1)=2$$

Evaluation Strategies (Problem-Solving Strategy 2.3):

1. **Factoring and Canceling:** Factor polynomials to cancel common factors.

   Example 2.17:

$$\lim_{x\to 3}\frac{x^2-9}{x-3}=\lim_{x\to 3}\frac{(x-3)(x+3)}{x-3}=\lim_{x\to 3}(x+3)=6.$$

   Evaluating a Limit by Factoring and Canceling

   Evaluate. $\lim_{x\to 3}\frac{x^2-3x}{2x^2-5x-3}$

   Solution

   Step 1. The function $f(x)=\frac{x^2-3x}{2x^2-5x-3}$ is undefined for $x=3$. In fact, if we substitute 3 into the function we get $0/0$, which is undefined. Factoring and canceling is a good strategy:

$$\lim_{x\to 3}\frac{x^2-3x}{2x^2-5x-3}=\lim_{x\to 3}\frac{x(x-3)}{(x-3)(2x+1)}$$

   Step 2. For all $x \neq 3$, $\frac{x^2-3x}{2x^2-5x-3}=\frac{x}{2x+1}$. Therefore,

$$\lim_{x\to 3}\frac{x(x-3)}{(x-3)(2x+1)}=\lim_{x\to 3}\frac{x}{2x+1}$$

   Step 3. Evaluate using the limit laws:

$$\lim_{x \to 3} \frac{x}{2x+1} = \frac{3}{7}$$

2. **Multiplying by the Conjugate:** Used if an expression involves a square root difference (e.g., $\sqrt{x+2}-1$).

Evaluating a Limit by Multiplying by a Conjugate

Evaluate $\lim\limits_{x \to -1} \frac{\sqrt{x+2}-1}{x+1}$

Solution

Step 1. $\frac{\sqrt{x+2}-1}{x+1}$ has the form $0/0$ at -1. Let's begin by multiplying by $\sqrt{x+2}+1$, the conjugate of $\sqrt{x+2}-1$, on the numerator and denominator:

$$\lim_{x \to -1} \frac{\sqrt{x+2}-1}{x+1} = \lim_{x \to -1} \frac{\sqrt{x+2}-1}{x+1} \cdot \frac{\sqrt{x+2}+1}{\sqrt{x+2}+1}$$

Step 2. We then multiply out the numerator. We don't multiply out the denominator because we are hoping that the $(x+1)$ in the denominator cancels out in the end:

$$= \lim_{x \to -1} \frac{x+1}{(x+1)(\sqrt{x+2}+1)}$$

Step 3. Then we cancel:

$$= \lim_{x \to -1} \frac{1}{\sqrt{x+2}+1}$$

Step 4. Last, we apply the limit laws:

$$\lim_{x \to -1} \frac{1}{\sqrt{x+2}+1} = \frac{1}{2}$$

3. **Simplifying Complex Fractions:** Simplify the fraction algebraically first.

Evaluating a Limit by Simplifying a Complex Fraction

Evaluate $\lim\limits_{x \to 1} \frac{\frac{1}{x+1}-\frac{1}{2}}{x-1}$

Solution

Step 1. $\frac{\frac{1}{x+1}-\frac{1}{2}}{x-1}$ has the form $0/0$ at 1. We simplify the algebraic fraction by multiplying by $2(x+1)/2(x+1)$:

$$\lim_{x \to 1} \frac{\frac{1}{x+1}-\frac{1}{2}}{x-1} = \lim_{x \to 1} \frac{\frac{1}{x+1}-\frac{1}{2}}{x-1} \cdot \frac{2(x+1)}{2(x+1)}$$

Step 2. Next, we multiply through the numerators. Do not multiply the denominators because we want to be able to cancel the factor $(x-1)$:

$$= \lim_{x \to 1} \frac{2-(x+1)}{2(x-1)(x+1)}$$

Step 3. Then, we simplify the numerator:

$$= \lim_{x \to 1} \frac{-x+1}{2(x-1)(x+1)}$$

Step 4. Now we factor out -1 from the numerator:

$$= \lim_{x \to 1} \frac{-(x-1)}{2(x-1)(x+1)}$$

Step 5. Then, we cancel the common factors of $(x-1)$:

$$= \lim_{x \to 1} \frac{-1}{2(x+1)}$$

Step 6. Last, we evaluate using the limit laws:

$$\lim_{x \to 1} \frac{-1}{2(x+1)} = -\frac{1}{4}$$

4. Evaluating a Two-Sided Limit Using the Limit Laws

For the piecewise function:

$$f(x) = \begin{cases} 4x-3 & \text{if } x < 2 \\ (x-3)^2 & \text{if } x \geq 2 \end{cases}$$

evaluate each of the following limits:

a. $\lim\limits_{x \to 2^-} f(x)$

b. $\lim\limits_{x \to 2^+} f(x)$

c. $\lim\limits_{x \to 2} f(x)$

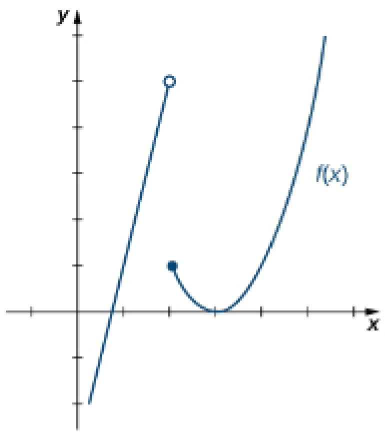


Figure 2.26 This graph shows a function $f(x)$.

Solution

a. When $x \to 2^-$ (i.e., x approaches 2 from values less than 2), the function is defined by $f(x) = 4x-3$. Using the limit laws:

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (4x-3) = 4 \cdot 2 - 3 = 5$$

b. When $x \to 2^+$ (i.e., x approaches 2 from values greater than or equal to 2), the function is defined by $f(x) = (x-3)^2$. Using the limit laws:

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x-3)^2 = (2-3)^2 = (-1)^2 = 1$$

c. For the two-sided limit $\lim_{x \to 2} f(x)$ to exist, the left-hand limit and right-hand limit must be equal. Since $\lim_{x \to 2^-} f(x) = 5$ and $\lim_{x \to 2^+} f(x) = 1$ are not equal:

$$\lim_{x \to 2} f(x) \quad \text{does not exist}$$

C. The Squeeze Theorem

Theorem 2.7 (The Squeeze Theorem): If $h(x) \leq f(x) \leq g(x)$ over an open interval containing a, and $\lim_{x \to a} h(x) = L$ and $\lim_{x \to a} g(x) = L$ (where L is a real number), then $\lim_{x \to a} f(x) = L$.

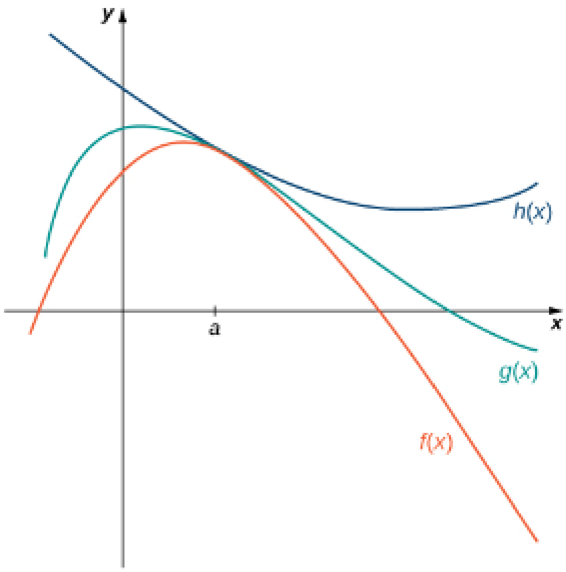


Figure 2.27 The Squeeze Theorem applies when $f(x) \leq g(x) \leq h(x)$ and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x)$.

Key Trigonometric Limit: This theorem is crucial for proving that $\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$.

Proof:

Squeeze Theorem Proof

1. Geometric Inequality (for $0 < \theta < \frac{\pi}{2}$)

In the unit circle:

$$\sin(\theta) \leq \theta \leq \tan(\theta)$$

Dividing by $\sin(\theta) > 0$:

$$1 \leq \frac{\theta}{\sin(\theta)} \leq \frac{1}{\cos(\theta)}$$

Taking reciprocals:

$$\cos(\theta) \leq \frac{\sin(\theta)}{\theta} \leq 1$$

2. Limits of Bounding Functions

$$\lim_{\theta \to 0} \cos(\theta) = 1$$

$$\lim_{\theta \to 0} 1 = 1$$

3. Squeeze Theorem Application

Since for all $\theta \in (0, \frac{\pi}{2})$:

$$\cos(\theta) \leq \frac{\sin(\theta)}{\theta} \leq 1$$

and

$$\lim_{\theta \to 0} \cos(\theta) = \lim_{\theta \to 0} 1 = 1$$

By the Squeeze Theorem:

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$$

4. Alternative Area Proof

Using sector area comparison:

$$\frac{1}{2} \sin(\theta) \cos(\theta) \leq \frac{1}{2} \theta \leq \frac{1}{2} \tan(\theta)$$

Multiplying by 2:

$$\sin(\theta) \cos(\theta) \leq \theta \leq \tan(\theta)$$

Dividing by $\sin(\theta)$:

$$\cos(\theta) \leq \frac{\theta}{\sin(\theta)} \leq \frac{1}{\cos(\theta)}$$

Taking reciprocals gives the same inequality.

Check the gif demo:

https://calc1.drbin.top/outputs/sin_theta_limit_english.gif

IV. Continuity (2.4)

Intuitive Definition: A function is **continuous** if its graph can be traced with a pencil without lifting it from the page.

A. Continuity at a Point

Definition: A function f is **continuous at a point** a if and only if three conditions are satisfied:

1. $f(a)$ **is defined**. (Prevents a hole/gap where the function value is missing).
2. $\lim_{x \to a} f(x)$ **exists**. (Prevents a jump or oscillation).
3. $\lim_{x \to a} f(x) = f(a)$. (Ensures the limit value matches the function value).

Example (Continuity Check):

Example 2.28: Determine if $f(x) = \frac{x^2-1}{x-1}$ for $x \neq 1$ and $f(1) = 2$ is continuous at $x = 1$.

1. $f(1) = 2$ (Defined).

Counterexample: **f(a) is not defined**.

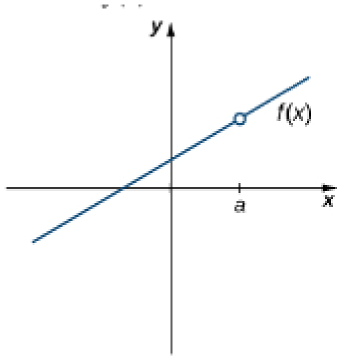


Figure 2.32 The function $f(x)$ is not continuous at a because $f(a)$ is undefined.

2. $\lim_{x \to 1} f(x) = 2$ (Exists, calculated by simplifying).

Counterexample: $\lim_{x \to a}$ **f(x) does not exist**.

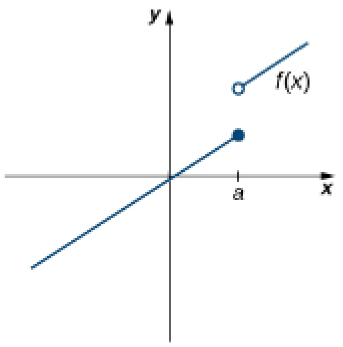


Figure 2.33 The function $f(x)$ is not continuous at a because $\lim\limits_{x \to a} f(x)$ does not exist.

3. $\lim_{x \to 1} f(x) = 2 = f(1)$. (Satisfied).

Counterexample: $\lim_{x \to a} f(x) \neq f(a)$.

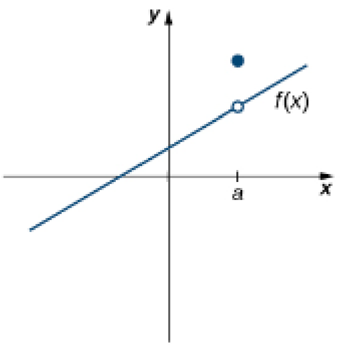


Figure 2.34 The function $f(x)$ is not continuous at a because $\lim\limits_{x \to a} f(x) \neq f(a)$.

Conclusion: f is continuous at 1.

B. Types of Discontinuities

If f is discontinuous at a, the discontinuity is classified:

1. **Removable Discontinuity:** $\lim_{x\to a} f(x)$ exists (is a real number L), but $f(a)$ is either undefined or $\lim_{x\to a} f(x) \neq f(a)$. (Intuitively, a "hole" in the graph). **Example 2.30:** $f(x) = \frac{x^2-4}{x-2}$ at $x = 2$. $\lim_{x\to 2} f(x) = 4$ exists, but $f(2)$ is undefined. This is a **removable discontinuity**.
2. **Jump Discontinuity:** Both $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist (as real numbers) but are unequal. **Example 2.31:** The piecewise function from Example 2.27 at $x = 3$. Since $\lim_{x\to 3^-} f(x)$ and $\lim_{x\to 3^+} f(x)$ exist but differ, it has a **jump discontinuity**.
3. **Infinite Discontinuity:** $\lim_{x\to a} f(x)$ or one of the one-sided limits is $\pm\infty$. (Occurs at a vertical asymptote). **Example 2.32:** $f(x) = 1/(x+1)$ at $x = -1$. Since $\lim_{x\to -1} f(x) = \pm\infty$, it has an **infinite discontinuity**.

C. The Intermediate Value Theorem (IVT)

Theorem 2.11 (IVT): If f is **continuous** over a closed, bounded interval $[a, b]$, and z is any real number between $f(a)$ and $f(b)$, then there exists a number c in $[a, b]$ satisfying $f(c) = z$.

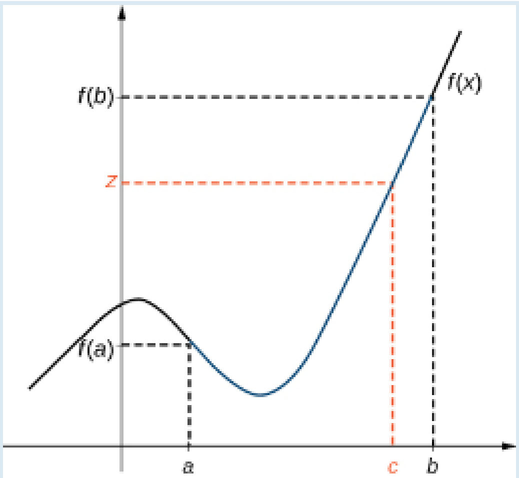


Figure 2.38 There is a number $c \in [a, b]$ that satisfies $f(c) = z$.

Key Application: Finding zeros (roots) of a function.

Example 2.36: Show $f(x) = x^3 - 4x + 2$ has at least one zero over $[0, 1]$. Since $f(x)$ is a polynomial, it is continuous. $f(0) = 2$ and $f(1) = 1^3 - 4(1) + 2 = -1$. Since $f(0) > 0$ and $f(1) < 0$, and $z = 0$ is between -1 and 2, the IVT guarantees a root c exists in $[0, 1]$ such that $f(c) = 0$.

Reminder (IVT Requirement): The function *must* be continuous over the closed interval $[a, b]$ for the theorem to apply.

V. The Precise Definition of a Limit (2.5)

This converts the intuitive idea of limits into a formal, rigorous mathematical definition ($\epsilon - \delta$ definition).

A. Quantifying Closeness

ϵ **(Epsilon):** Represents the positive distance from the limit L. $|f(x) - L| < \epsilon$ means $f(x)$ is closer than ϵ to L. δ **(Delta):** Represents the positive distance from a. $0 < |x - a| < \delta$ means x is closer than δ to a, and $x \neq a$.

B. The Epsilon-Delta Definition

Definition: $\lim_{x \to a} f(x) = L$ if, **for every** $\epsilon > 0$**, there exists a** $\delta > 0$ such that **if** $0 < |x - a| < \delta$**, then** $|f(x) - L| < \epsilon$.

Part of Definition	Interpretation	Citation
For every $\epsilon > 0$	Universal Quantifier: For every positive distance ϵ from L.	
There exists a $\delta > 0$	Existential Quantifier: There is a positive distance δ from a.	

Tool for Proofs: The **Triangle Inequality** states that for any real numbers a and b, $|a + b| \leq |a| + |b|$.

Example (Linear Proof - Example 2.39): Statement to prove: $\lim_{x \to 3}(2x + 1) = 7$. **Proof Sketch (Finding** δ**):** We start with $|f(x) - L| < \epsilon$: $|(2x + 1) - 7| < \epsilon \Rightarrow |2x - 6| < \epsilon \Rightarrow 2|x - 3| < \epsilon \Rightarrow |x - 3| < \epsilon/2$

Choose: $\delta = \epsilon/2$. **Conclusion:** If $0 < |x - 3| < \epsilon/2$, then $|(2x + 1) - 7| = 2|x - 3| < 2(\epsilon/2) = \epsilon$. Therefore, the limit is 7.

C. Precise Definition of Infinite Limits

The $\epsilon - \delta$ definition can be modified for infinite limits using M (an arbitrarily large positive number) instead of ϵ.

Definition (∞**):** $\lim_{x \to a} f(x) = \infty$ if, **for every** $M > 0$, there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then $f(x) > M$.

Definition ($-\infty$**):** $\lim_{x \to a} f(x) = -\infty$ if, **for every** $N < 0$, there exists $\delta > 0$ such that if $0 < |x - a| < \delta$, then $f(x) < N$.